# MANY-PARAMETER M-COMPLEMENTARY GOLAY SEQUENCES AND TRANSFORMS

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**Abstract**

In this paper, we develop the family of Golay–Rudin–Shapiro (GRS) m-complementary many-parameter sequences and many-parameter Golay transforms. The approach is based on a new generalized iteration generating construction, associated with n unitary many-parameter transforms and n arbitrary groups of given fixed order. We are going to use multi-parameter Golay transform in Intelligent-OFDM-TCS instead of discrete Fourier transform in order to find out optimal values of parameters optimized PARP, BER, SER, anti-eavesdropping and anti-jamming effects.

Keywords: complementary sequences, many-parameter orthogonal transforms, fast algorithms, OFDM systems.

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## Introduction

Binary $\pm 1$-valued *Golay – Rudin – Shapiro* sequences (2-GRSS) associated with the cyclic group $\mathbf{Z}_2^n$ were introduced independently by Golay [1, 2, 3] in 1949-1951, Shapiro [4, 5] and Rudin [6] in 1951.M.J.E.Golay [2] introduced the general concept of "complementary pairs" of finite sequences all of whose entries are $\pm 1$. For building the classical FGRST in bases of classical 2-GRSS the following actors are used: 1) Abelian group $\mathbf{Z}_2$, 2) 2-point Fourier transform $\mathcal{F}_2$, and 3) complex field $\mathbf{C}$, *i.e.,* these transforms are associated with the triple $(\mathbf{Z}_2, \mathcal{F}_2, \mathbf{C})$.

In previous papers [7, 8], we have shown a new unified approach to the $\mathbf{GF}(p)$ -, or Clifford-valued complementary sequences and Golay transforms. It was associated not with the triple $(\mathbf{Z}_2, \mathcal{F}_2, \mathbf{C})$, but with triples

$$(\mathbf{Z}_2\,, \{\mathcal{CS}_2^1(\varphi_1,\alpha_1,\gamma_1), \mathcal{CS}_2^2(\varphi_2,\alpha_2,\gamma_2),..., \mathcal{CS}_n^1(\varphi_n,\alpha_n,\gamma_n)\}, \mathcal{A}lg)$$

and $\left(\mathbf{Z}_2, \mathcal{CS}_2(\varphi,\alpha,\gamma), \mathcal{A}lg\right)$, where $\{\mathcal{CS}_2^1(\varphi_1,\alpha_1,\gamma_1), \mathcal{CS}_2^2(\varphi_2,\alpha_2,\gamma_2),...,\mathcal{CS}_n^1(\varphi_n,\alpha_n,\gamma_n)\}$ is a set of arbitrary unitary(2×2) -transforms of type

$$\mathcal{CS}_2(\varphi_k,\alpha_k,\gamma_k) = \begin{bmatrix} e^{i\alpha_k}\cos\varphi_k & e^{i\gamma_k}\sin\varphi_k \\ e^{-i\gamma_k}\sin\varphi_k & -e^{-i\alpha_k}\cos\varphi_k \end{bmatrix},$$

$$k = 1,...,n,$$

and $\mathcal{CS}_2(\varphi,\alpha,\gamma)$ is a single transform, $\mathcal{A}lg$ is an algebra (for example, Clifford algebra).

In this work, we develop a new unified approach to the so-called generalized multi-parameter $m-$complementary sequences. This construction has a rich algebraic structure. It is associated not with the triple $(\mathbf{Z}_2, \mathcal{F}_2, \mathbf{C})$, but with

1) $\left(\mathbf{Z}_m, \mathbf{U}_m, \mathcal{A}lg\right)$, 2) $\left(\mathbf{Z}_m, \{\mathbf{U}_m^1, \mathbf{U}_m^2,...,\mathbf{U}_m^n\}, \mathcal{A}lg\right)$,

3) $\left(\mathbf{Gr}_m, \{\mathbf{U}_m^1, \mathbf{U}_m^2,...,\mathbf{U}_m^n\}, \mathcal{A}lg\right)$,

4) $\left(\{\mathbf{Gr}_m^1, \mathbf{Gr}_m^2,...,\mathbf{Gr}_m^n\}, \{\mathbf{U}_m^1, \mathbf{U}_m^2,...,\mathbf{U}_m^n\}, \mathcal{A}lg\right)$.

where $\{\mathbf{Gr}_m^1, \mathbf{Gr}_m^2,...,\mathbf{Gr}_m^n\}$ is a set of arbitrary finite groups of given order $m$ Here $\{\mathbf{U}_m^1, \mathbf{U}_m^2,...,\mathbf{U}_m^n\}$ is a set of arbitrary unitary ($m$×$m$) – transforms represented in the many-parameter Jacobi-Euler form [9 – 10]:

$$\mathbf{U}_m^1 = \mathbf{U}_m^1(\varphi_0^1,\varphi_1^1,...,\varphi_q^1) = \mathbf{U}_m^1(\boldsymbol{\varphi}_q^1) = \prod_{r=1}^{m-1}\prod_{s=r+1}^{m}\mathbf{J}(\varphi_{r,s}^1),$$

$$\mathbf{U}_m^2 = \mathbf{U}_m^2(\varphi_0^2,\varphi_1^2,...,\varphi_q^2) = \mathbf{U}_m^2(\boldsymbol{\varphi}_q^2) = \prod_{r=1}^{m-1}\prod_{s=r+1}^{m}\mathbf{J}(\varphi_{r,s}^2),$$

$$...,$$

$$\mathbf{U}_m^n = \mathbf{U}_m^n(\varphi_0^n,\varphi_1^n,...,\varphi_q^n) = \mathbf{U}_m^n(\boldsymbol{\varphi}_q^n) = \prod_{r=1}^{m-1}\prod_{s=r+1}^{m}\mathbf{J}(\varphi_{r,s}^n),$$

where

$$\mathbf{J}(\varphi_{r,s}) = \begin{array}{c} \\ \\ \\ r \\ \\ s \\ \\ \\ \end{array} \begin{array}{c} \qquad\quad r \qquad\qquad\qquad s \\ \left(\begin{array}{cc:c:c:c}
1 & \cdots & 0 & \cdots & 0 \ \cdots \ 0 \\
\vdots & \ddots & \vdots & & \vdots \quad\ \ \vdots \\ \hdashline
0 & \cdots & c(\varphi_{r,s}) & \cdots & s(\varphi_{r,s}) \ \cdots \ 0 \\ \hdashline
\vdots & & \vdots & \ddots & \vdots \quad\ \ \vdots \\ \hdashline
0 & \cdots & s(\varphi_{r,s}) & \cdots & -c(\varphi_{r,s}) \ \cdots \ 0 \\ \hdashline
\vdots & & \vdots & & \vdots \quad \ddots \ \vdots \\
0 & \cdots & 0 & \cdots & 0 \ \cdots \ 1
\end{array}\right) \end{array},$$

is the Jacobi orthonormal rotation with reflection, $\boldsymbol{\varphi}_q^1 = (\varphi_0^1,\varphi_1^1,...,\varphi_q^1),...,\boldsymbol{\varphi}_q^n = (\varphi_0^n,\varphi_1^n,...,\varphi_q^n)$ are the Jacobi parameters, $q = C_m^2 = m(m-1)/2$, $c(\varphi_{r,s}) = \cos(\varphi_{r,s})$, $s(\varphi_{r,s}) = \sin(\varphi_{r,s})$.

The rest of the paper is organized as follows: in Section 2, the object of the study (*Golay – Rudin – Shapiro m*-ary sequences) is described. In Section 3 we propose method based on new generalized iteration rule with $n$ unitary ($m$×$m$)-transforms $\mathbf{U}_m^1, \mathbf{U}_m^2,...,\mathbf{U}_m^n$ and single group $\mathbf{Z}_m$. Then we generalize the previously method on $n$ unitary ($m$×$m$)-transforms $\mathbf{U}_m^1, \mathbf{U}_m^2,...,\mathbf{U}_m^n$ and on $n$ finite groups $\{\mathbf{Gr}_m^1, \mathbf{Gr}_m^2,...,\mathbf{Gr}_m^n\}$. In Section 5 we derive fast algorithms for binary Golay transforms.

### *The object of the study.*
### *New iteration construction for original Golay sequences*

We begin by describing the original Golay $m$-complementary sequences.

**Definition 1**. *A generalization of the Golay complementary pair, known as the Golay m-Complementary m-element Set (m-*GCS*) of complex-valued sequences* [11]

$$m\text{-GCS}=\begin{cases}\mathrm{com}_0(t) := \left(c_0(0),\ c_0(1),\ \ldots,\ c_0(m-1)\right),\\ \mathrm{com}_1(t) := \left(c_1(0),\ c_1(1),\ \ldots,\ c_1(m-1)\right),\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\ \mathrm{com}_{m-1}(t) := \left(c_{m-1}(0), c_{m-1}(1), \ldots, c_{m-1}(m-1)\right)\end{cases}$$

is defined by $\sum_{k=0}^{m-1} COR_k(\tau) = m\cdot\delta(\tau),\ \sum_{k=0}^{m-1}\left|\mathrm{COM}_1(z)\right|^2 = m,$ where $\left\{COR_k(\tau)\right\}_{k=0}^{m-1}$ are the periodic autocorrelation functions of $\left\{\mathrm{com}_k(t)\right\}_{k=1}^{m}$ and $\mathrm{COM}_k(z) = \mathcal{Z}\left\{\mathrm{com}_k(t)\right\}$ are their $\mathcal{Z}$ – transforms.

We use two symbols $\boldsymbol{\alpha}_n \in [0, m^{n-1}-1] = \mathbf{Z}_{m^n}$ and $\mathbf{t}_n \in [0, m^{n-1}-1] = \mathbf{Z}_{m^n}$ for numeration of Golay sequences and discrete time, respectively. For integer $\boldsymbol{\alpha}_n \in [0, m^{n-1}-1] = \mathbf{Z}_{m^n}$ and $\mathbf{t}_n \in [0, m^{n-1}-1] = \mathbf{Z}_{m^n}$ we shall use m-arycodes $\vec{\boldsymbol{\alpha}}_n = \left(\alpha_1, \alpha_2, \ldots, \alpha_n\right)$, $\vec{\mathbf{t}}_n = \left(t_1, t_2, \ldots, t_n\right)$, where $\alpha_i t_i \in \{0, 1, \ldots, m-1\} = \mathbf{Z}_m$, $i = 1, 2, \ldots, n$.

*Let* $\vec{\boldsymbol{\alpha}}_n = \left(\alpha_1, \alpha_2, \ldots, \alpha_n\right)$ *and* $\vec{\mathbf{t}}_n = \left(t_1, t_2, \ldots, t_n\right)$ *be* m-*ary codes, then define*

$$\boldsymbol{\alpha}_n = \left|\vec{\boldsymbol{\alpha}}_n\right| = \sum_{i=1}^{n}\alpha_{n-i+1}m^{i-1},\ \text{and}\ \mathbf{t}_n = \left|\vec{\mathbf{t}}_n\right| = \sum_{i=1}^{n} t_{n-i+1}m^{n-i}$$

as integers whose m-ary codes are $\vec{\boldsymbol{\alpha}}_n = \left(\alpha_1, \alpha_2, \ldots, \alpha_n\right)$ and $\vec{\mathbf{t}}_n = \left(t_1, t_2, \ldots, t_n\right)$, where $\alpha_n, t_1$ are less significant bits (LSB) and $\alpha_1, t_n$ are most significant bits (MSB) of $\vec{\boldsymbol{\alpha}}_n = \left(\alpha_1, \alpha_2, \ldots, \alpha_n\right)$ and $\vec{\mathbf{t}}_n = \left(t_1, t_2, \ldots, t_n\right)$, respectively. Obviously,

$$\begin{array}{ll}
\vec{\boldsymbol{\alpha}}_1 = (\alpha_1) \in \mathbf{Z}_m, & \boldsymbol{\alpha}_1 = \alpha_1 \in \mathbf{Z}_m,\\
\vec{\boldsymbol{\alpha}}_2 = \left(\vec{\boldsymbol{\alpha}}_1, \alpha_2\right) \in \mathbf{Z}_m \times \mathbf{Z}_m = \mathbf{Z}_m^2, & \left(\boldsymbol{\alpha}_1, \alpha_2\right) \in \mathbf{Z}_m \times \mathbf{Z}_m,\\
\vec{\boldsymbol{\alpha}}_3 = \left(\vec{\boldsymbol{\alpha}}_2, \alpha_3\right) \in \mathbf{Z}_m^2 \times \mathbf{Z}_m = \mathbf{Z}_m^3, & \left(\boldsymbol{\alpha}_2, \alpha_3\right) \in \mathbf{Z}_{m^2} \times \mathbf{Z}_m,\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, & \ldots\ldots\ldots\ldots\ldots\ldots,\\
\vec{\boldsymbol{\alpha}}_n = \left(\vec{\boldsymbol{\alpha}}_{n-1}, \alpha_n\right) \in \mathbf{Z}_m^{n-1} \times \mathbf{Z}_m = \mathbf{Z}_m^n, & \left(\boldsymbol{\alpha}_{n-1}, \alpha_n\right) \in \mathbf{Z}_{m^{n-1}}^1 \times \mathbf{Z}_m;\\
\vec{\mathbf{t}}_1 = \left(t_1\right) \in \mathbf{Z}_m, & \mathbf{t}_1 = t_1 \in \mathbf{Z}_m,\\
\vec{\mathbf{t}}_2 = \left(\vec{\mathbf{t}}_1, t_2\right) \in \mathbf{Z}_m \times \mathbf{Z}_m = \mathbf{Z}_m^2, & \left(\mathbf{t}_1, t_2\right) \in \mathbf{Z}_m \times \mathbf{Z}_m,\\
\vec{\mathbf{t}}_3 = \left(\vec{\mathbf{t}}_2, t_3\right) \in \mathbf{Z}_m^2 \times \mathbf{Z}_m = \mathbf{Z}_m^3, & \left(\mathbf{t}_2, t_3\right) \in \mathbf{Z}_{m^2} \times \mathbf{Z}_m,\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, & \ldots\ldots\ldots\ldots\ldots\ldots,\\
\vec{\mathbf{t}}_n = \left(\vec{\mathbf{t}}_{n-1}, t_n\right) \in \mathbf{Z}_m^{n-1} \times \mathbf{Z}_m = \mathbf{Z}_m^n, & \left(\mathbf{t}_{n-1}, t_n\right) \in \mathbf{Z}_{m^{n-1}} \times \mathbf{Z}_m.
\end{array}$$

Let $\left\{\mathrm{com}_{\boldsymbol{\alpha}_{n+1}}^{[n+1]}(\mathbf{t}_{n+1})\right\}$ be $m^{n+1}$-element set of $m$ complementary sequences (of length $m^{n+1}$), where $\boldsymbol{\alpha}_{n+1}, \mathbf{t}_{n+1} = 0, 1, .., m^{n+1}-1$ They form rows of a $(m^{n+1}\times m^{n+1})$ -matrix $\mathbf{G}_{m^{n+1}}^{[n+1]} = \left[\mathrm{com}_{\boldsymbol{\alpha}_{n+1}}^{[n+1]}(\mathbf{t}_{n+1})\right]_{\boldsymbol{\alpha}_{n+1}, \mathbf{t}_{n+1}=0}^{m^{n+1}-1}$, that is called *the m-Golay matrix.* Here index $[n+1]$ shows that Golay matrix have been obtained on the $n+1$ iteration step. We are going to group these rows (sequences) as

$$\mathbf{G}_{m^{n+1}}^{[n+1]} = \mathop{\boxplus}_{\boldsymbol{\alpha}_{n+1}=0}^{m^{n+1}-1}\mathrm{com}_{(\boldsymbol{\alpha}_{n+1})}^{[n+1]}(\mathbf{t}_{n+1}) = \mathop{\boxplus}_{\boldsymbol{\alpha}_n=0}^{m^n-1}\left(\mathop{\boxplus}_{\alpha_{n+1}=0}^{m-1}\mathrm{com}_{(\boldsymbol{\alpha}_n, \alpha_{n+1})}^{[n+1]}(\mathbf{t}_{n+1})\right) = \mathop{\boxplus}_{\boldsymbol{\alpha}_n=0}^{m^n-1}\begin{bmatrix}\mathrm{com}_{(\boldsymbol{\alpha}_n,0)}^{[n+1]}(\mathbf{t}_{n+1})\\ \mathrm{com}_{(\boldsymbol{\alpha}_n,1)}^{[n+1]}(\mathbf{t}_{n+1})\\ \ldots\\ \mathrm{com}_{(\boldsymbol{\alpha}_n,m-1)}^{[n+1]}(\mathbf{t}_{n+1})\end{bmatrix}. \quad (1)$$

Let us to select the more fine structure of the $m$-Golay matrix:

$$\mathbf{G}_{m^{n+1}}^{[n+1]} = \mathop{\boxplus}_{\boldsymbol{\alpha}_{n+1}=0}^{m^{n+1}-1}\mathrm{com}_{(\boldsymbol{\alpha}_{n+1})}^{[n+1]}(\mathbf{t}_{n+1}) = \mathop{\boxplus}_{\boldsymbol{\alpha}_n=\mathbf{0}}^{\mathbf{m^n-1}}\begin{bmatrix}\mathrm{com}_{(\boldsymbol{\alpha}_n,0)}^{[n+1]}(\mathbf{t}_{n+1})\\ \mathrm{com}_{(\boldsymbol{\alpha}_n,1)}^{[n+1]}(\mathbf{t}_{n+1})\\ \ldots\ldots\ldots\ldots\\ \mathrm{com}_{(\boldsymbol{\alpha}_n,m-1)}^{[n+1]}(\mathbf{t}_{n+1})\end{bmatrix} = \mathop{\boxplus}_{\boldsymbol{\alpha}_{n-1}=0}^{m^{n-1}-1}\left(\mathop{\boxplus}_{\alpha_n=0}^{m-1}\begin{bmatrix}\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},\alpha_n,0)}^{[n+1]}(\mathbf{t}_{n+1})\\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},\alpha_n 1)}^{[n+1]}(\mathbf{t}_{n+1})\\ \ldots\ldots\ldots\ldots\\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},\alpha_n m-1)}^{[n+1]}(\mathbf{t}_{n+1})\end{bmatrix}\right) = \mathop{\boxplus}_{\boldsymbol{\alpha}_{n-1}=0}^{m^{n-1}-1}\left[\begin{array}{c}\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0,0)}^{[n+1]}(\mathbf{t}_{n+1})\\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0,1)}^{[n+1]}(\mathbf{t}_{n+1})\\ \ldots\\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0,m-1)}^{[n+1]}(\mathbf{t}_{n+1})\\ \hline \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1,0)}^{[n+1]}(\mathbf{t}_{n+1})\\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1,1)}^{[n+1]}(\mathbf{t}_{n+1})\\ \ldots\\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1,m-1)}^{[n+1]}(\mathbf{t}_{n+1})\\ \hline \vdots\\ \vdots\\ \hline \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},m-1,0)}^{[n+1]}(\mathbf{t}_{n+1})\\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},m-11)}^{[n+1]}(\mathbf{t}_{n+1})\\ \ldots\\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},m-1m-1)}^{[n+1]}(\mathbf{t}_{n+1})\end{array}\right]. \quad (2)$$

**Example 1.** For $n=1$ and $n=2$ we have, respectively,

$$\mathbf{G}^{[1]}_{3^1}=\left[\mathrm{com}^{[1]}_{\boldsymbol{\alpha}_1}(\mathbf{t}_1)\right]^2_{\boldsymbol{\alpha}_1,\mathbf{t}_1=0}=\boxplus_{\boldsymbol{\alpha}_1=0}^{2}\mathrm{com}^{[1]}_{\boldsymbol{\alpha}_1}(\mathbf{t}_1)=\begin{bmatrix}\mathrm{com}^{[1]}_{(0)}(\mathbf{t}_1)\\ \mathrm{com}^{[1]}_{(1)}(\mathbf{t}_1)\\ \mathrm{com}^{[1]}_{(2)}(\mathbf{t}_1)\end{bmatrix},$$

$$\mathbf{G}^{[2]}_{3^2}=\boxplus_{\boldsymbol{\alpha}_1=0}^{2}\begin{bmatrix}\mathrm{com}^{[2]}_{(\boldsymbol{\alpha}_1,0)}(\mathbf{t}_2)\\ \mathrm{com}^{[2]}_{(\boldsymbol{\alpha}_1,1)}(\mathbf{t}_2)\\ \mathrm{com}^{[2]}_{(\boldsymbol{\alpha}_1,2)}(\mathbf{t}_2)\end{bmatrix}=\begin{bmatrix}\mathrm{com}^{[2]}_{(0,0)}(\mathbf{t}_2)\\ \mathrm{com}^{[2]}_{(0,1)}(\mathbf{t}_2)\\ \mathrm{com}^{[2]}_{(0,2)}(\mathbf{t}_2)\\ \hline \mathrm{com}^{[2]}_{(1,0)}(\mathbf{t}_2)\\ \mathrm{com}^{[2]}_{(1,1)}(\mathbf{t}_2)\\ \mathrm{com}^{[2]}_{(1,2)}(\mathbf{t}_2)\\ \hline \mathrm{com}^{[2]}_{(2,0)}(\mathbf{t}_2)\\ \mathrm{com}^{[2]}_{(2,1)}(\mathbf{t}_2)\\ \mathrm{com}_{(2,2)}(\mathbf{t}_2)\end{bmatrix}. \square$$

The matrix $\mathbf{G}^{[n+1]}_{m^{n+1}}$ is constructed by an iteration construction. The initial matrix $\mathbf{G}^{[1]}_{m^1}$ is formed by starting with an arbitrary unitary $(m\times m)$-matrix (in many-parameter form or not)

$$\mathbf{U}_m=\left[A_\alpha(t)\right]:=\mathbf{G}^{[1]}_{m^1}=\begin{bmatrix}\mathrm{com}^{[1]}_0(\mathbf{t}_1)\\ \mathrm{com}^{[1]}_1(\mathbf{t}_1)\\ \dots\dots\dots\dots\\ \mathrm{com}^{[1]}_{m-1}(\mathbf{t}_1)\end{bmatrix}=$$

$$=\begin{bmatrix}A_0(0) & A_0(1) & A_0(2) & \dots & A_0(m-1)\\ A_1(0) & A_1(1) & A_1(2) & \dots & A_1(m-1)\\ A_2(0) & A_2(1) & A_2(2) & \dots & A_2(m-1)\\ \dots & \dots & \dots & \dots & \dots\\ A_{m-1}(0) & A_{m-1}(1) & A_{m-1}(2) & \dots & A_{m-1}(m-1)\end{bmatrix},$$

**where** $A_\alpha(t)\in\mathcal{A}lg$,

$\mathrm{com}^{[1]}_\alpha(t)=\left(A_\alpha(0),\ A_\alpha(1),\dots,\ A_\alpha(m-1)\right)$.

**Example 2**. The initial matrix $\mathbf{G}^{[1]}_{m^1}$ can be the Fourier transform on Abelian group $\mathbf{Z}_m$:

$$\mathbf{G}^{[1]}_{m^1}=\begin{bmatrix}\mathrm{com}^{[1]}_0(\mathbf{t}_1)\\ \mathrm{com}^{[1]}_1(\mathbf{t}_1)\\ \mathrm{com}^{[1]}_2(\mathbf{t}_1)\\ \dots\dots\dots\dots\\ \mathrm{com}^{[1]}_{m-1}(\mathbf{t}_1)\end{bmatrix}=$$
$$=\begin{bmatrix}1 & 1 & 1 & \dots & 1\\ 1 & \varepsilon^{1\cdot1} & \varepsilon^{1\cdot2} & \dots & \varepsilon^{1\cdot(m-1)}\\ 1 & \varepsilon^{2\cdot1} & \varepsilon^{2\cdot2} & \dots & \varepsilon^{2\cdot(m-1)}\\ \dots & \dots & \dots & \dots & \dots\\ 1 & \varepsilon^{(m-1)\cdot1} & \varepsilon^{(m-1)\cdot2} & \dots & \varepsilon^{(m-1)\cdot(m-1)}\end{bmatrix}, \quad (3)$$

where $\varepsilon_m=\sqrt[m]{1}\in\mathcal{A}lg$, $\mathrm{com}^{[1]}_k(\mathbf{t})=\left(1,\varepsilon^{k\cdot1},\varepsilon^{k\cdot2},\dots,\varepsilon^{k\cdot(m-1)}\right)$, $(k=0,1,\dots,m-1)$ are characters $\mathbf{Z}_m$. □

It is easy to check that

$$\left(\left|\mathrm{COM}_0(z)\right|^2+\left|\mathrm{COM}_1(z)\right|^2+\dots+\left|\mathrm{COM}_{m-1}(z)\right|^2\right)_{|z|=1}=m.$$

**Indeed,**

$$\sum_{k=1}^{m-1}\left|\mathrm{COM}_k(z)\right|^2=\sum_{k=1}^{m-1}\mathrm{COM}_k(z)\overline{\mathrm{COM}}^k(\overline{z})=$$
$$=\sum_{k=1}^{m-1}\left(\sum_{t=0}^{m-1}a_k(t)z^t\right)\left(\sum_{s=0}^{m-1}\overline{a}_k(s)\overline{z}^s\right)=$$
$$=\sum_{s=0}^{m-1}\sum_{t=0}^{m-1}\left(\sum_{k=0}^{m-1}a_k(t)\overline{a}_k(s)\right)z^t\overline{z}^s=\sum_{s=0}^{m-1}\sum_{t=0}^{m-1}\delta_{t-s}z^t\overline{z}^s=\sum_{t=0}^{m-1}|z|^{2t},$$

since $\sum_{k=0}^{m-1}a_k(t)\overline{a}_k(s)=\delta_{t-s}$ is true for an arbitrary unitary (orthogonal) matrix. Hence,

$$\left(\sum_{k=1}^{m-1}\left|\mathrm{COM}_k(z)\right|^2\right)\Bigg|_{|z|=1}=\left(\sum_{t=0}^{m-1}|z|^{2t}\right)_{|z|=1}=m$$

and initial sequences in the form of rows of an unitary matrix (in particular case, in the form of characters $\mathrm{com}_k(\mathbf{t}_1)=(1,\varepsilon^{k\cdot1},\varepsilon^{k\cdot2},\dots,\varepsilon^{k\cdot(m-1)})$ of cyclic group $\mathbf{Z}_m$) are the Golay $m$-complementary sequences.

### *Methods*

The matrix $\mathbf{G}^{[n+1]}_{m^{n+1}}$ is constructed by an iteration construction

$$\mathbf{G}^{[1]}_{m^1}(\mathbf{U}^1_m)\xrightarrow{\mathbf{U}^2_m}\mathbf{G}^{[2]}_{m^2}(\mathbf{U}^1_m,\mathbf{U}^2_m)\xrightarrow{\mathbf{U}^3_m}\dots\xrightarrow{\mathbf{U}^{n+1}_m}\mathbf{G}^{[n+1]}_{m^{n+1}}(\mathbf{U}^1_m,\dots,\mathbf{U}^n_m,\mathbf{U}^{n+1}_m), \quad (4)$$

where

$$\mathcal{U}_{n+1}:=\left\{\mathbf{U}^1_m,\dots,\mathbf{U}^n_m,\mathbf{U}^{n+1}_m\right\}=\left\{\mathcal{U}_n,\mathbf{U}^{n+1}_m\right\},$$
$$\mathcal{U}_n:=\left\{\mathbf{U}^1_m,\dots,\mathbf{U}^n_m\right\}.$$

Here $\mathbf{U}^s_m(\boldsymbol{\varphi}^s_q)=\left[A^s_\alpha(t\mid\boldsymbol{\varphi}^s_q)\right]^{m-1}_{\alpha,t=0}\in SU(\mathcal{A}lg,m)$ $(s=1,2,\dots,n)$ are a sequence of unitary many-parameter $(m\times m)$ -transforms, belonging to the special unitary group $SU(\mathcal{A}lg,m)$, where $s=1,2,\dots,n+1$ and $A^s_\alpha(t\mid\boldsymbol{\varphi}^s_q)$ are $\mathcal{A}lg$-valued many-parameter sequences.

Let us assume that we have $m$-Golay matrix $\mathbf{G}^{[n]}_{m^n}(\mathbf{U}_1,\dots,\mathbf{U}_n)=\mathbf{G}^{[n]}_{m^n}(\mathcal{U}_n)$ (depending on $n$ previous transforms $\mathbf{U}^1_m,\dots,\mathbf{U}^n_m$). We need to construct the next $m$-Golay matrix $\mathbf{G}^{[n+1]}_{m^{n+1}}(\mathbf{U}^1_m,\dots,\mathbf{U}^{n+1}_m)=\mathbf{G}^{[n+1]}_{m^{n+1}}(\mathcal{U}_{n+1})$ using only $\mathbf{G}^{[n]}_{m^n}(\mathbf{U}^1_m,\dots,\mathbf{U}^n_m)$ and $\mathbf{U}^{n+1}_m$. We are going to use for $m$-Golay matrix $\mathbf{G}^{[n]}_{m^n}(\mathcal{U}_n)$ the same structure as in (1):

$$\mathbf{G}^{[n]}_{m^n}(\mathcal{U}_n)=\boxplus_{\boldsymbol{\alpha}_n=0}^{m^n-1}\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_n)}(\mathbf{t}_n\mid\mathcal{U}_n)=$$
$$=\boxplus_{\boldsymbol{\alpha}_n=\mathbf{0}}^{\mathbf{m^n-1}}\begin{bmatrix}\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n\mid\mathcal{U}_n)\\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n\mid\mathcal{U}_n)\\ \dots\dots\dots\dots\\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},m-1)}(\mathbf{t}_n\mid\mathcal{U}_n)\end{bmatrix}. \quad (5)$$

For constructing $\mathbf{G}^{[n+1]}_{m^{n+1}}(\mathcal{U}_{n+1})$ from $\mathbf{G}^{[n]}_{m^n}(\mathcal{U}_n)$ we take each complementary set in the form

$$m\text{-GCS}^{[n]}\left(\mathcal{U}_n\right)=\begin{bmatrix}\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n\mid\mathcal{U}_n)\\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n\mid\mathcal{U}_n)\\ \dots\dots\dots\dots\\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},m-1)}(\mathbf{t}_n\mid\mathcal{U}_n)\end{bmatrix}$$

and construct $m$ shifted versa of their components

$$m\text{-GCS}^{[n]}\left(\mathcal{U}_n\right)\rightarrow\begin{array}{l}\nearrow\; m\text{-GCS}^{[n]}_{\alpha_n=0}\left(\mathcal{U}_{n+1}\right),\\ \quad\; m\text{-GCS}^{[n]}_{\alpha_n=1}\left(\mathcal{U}_{n+1}\right),\\ \quad\quad\dots\dots\dots\\ \searrow\; m\text{-GCS}^{[n]}_{\alpha_n=m-1}\left(\mathcal{U}_{n+1}\right),\end{array}$$

where

$$m\text{-GCS}^{[n]}_{\alpha_n}\left(\mathcal{U}_{n+1}\right)=\mathbf{U}^{n+1}_m\left(\mathbf{P}^{\alpha_n}_m\begin{bmatrix}\mathbf{I}_{\mathbf{t}_n}&&&\\&\mathbf{T}^{1\cdot m^n}_{\mathbf{t}_n}&&\\&&\ddots&\\&&&\mathbf{T}^{(m-1)\cdot m^n}_{\mathbf{t}_n}\end{bmatrix}\tilde{\mathbf{P}}^{\alpha_n}_m\right)\times$$
$$\times\begin{bmatrix}\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n\mid\mathcal{U}_n)\\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n\mid\mathcal{U}_n)\\ \dots\dots\dots\dots\\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},m-1)}(\mathbf{t}_n\mid\mathcal{U}_n)\end{bmatrix}\equiv\begin{bmatrix}\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n},0)}(\mathbf{t}_{n+1}\mid\mathcal{U}_{n+1})\\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n},1)}(\mathbf{t}_{n+1}\mid\mathcal{U}_{n+1})\\ \dots\dots\dots\dots\\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n},m-1)}(\mathbf{t}_{n+1}\mid\mathcal{U}_{n+1})\end{bmatrix}. \qquad (6)$$

Here $\alpha_n=0,1,\dots,m-1$, $\mathbf{P}^{\alpha_n}_m$ is the cyclic permutation operator on $\alpha_n$ positions (modulo $m$), $\mathbf{T}^{\mathbf{m^n}s}_{\mathbf{t}_n}$ is the shift operator on $m^n s$ positions $\mathbf{T}^{m^n s}_{\mathbf{t}_n}f(\mathbf{t}_n):=f(\mathbf{t}_n+m^n\,\mathrm{s})$, $\tilde{\mathbf{P}}_m$ is transposed matrix of $\mathbf{P}_m$.

According to (1) we obtain

$$\mathbf{G}^{[n+1]}_{m^{n+1}}(\mathcal{U}_{n+1})=\mathop{\boxplus}_{\alpha_n=0}^{m^n-1}\begin{bmatrix}\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n},0)}(\mathbf{t}_{n+1}\mid\mathcal{U}_{n+1})\\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n},1)}(\mathbf{t}_{n+1}\mid\mathcal{U}_{n+1})\\ \dots\dots\dots\dots\\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n},m-1)}(\mathbf{t}_{n+1}\mid\mathcal{U}_{n+1})\end{bmatrix}=$$
$$=\mathop{\boxplus}_{\alpha_n=0}^{m^n-1}\mathbf{U}^{n+1}_m\cdot\left(\mathbf{P}^{\alpha_n}_m\cdot\begin{bmatrix}\mathbf{I}_{\mathbf{t}_n}&&&\\&\mathbf{T}^{1\cdot m^n}_{\mathbf{t}_n}&&\\&&\ddots&\\&&&\mathbf{T}^{(m-1)\cdot m^n}_{\mathbf{t}_n}\end{bmatrix}\cdot\tilde{\mathbf{P}}^{\alpha_n}_m\right)\times \qquad (7)$$
$$\times\begin{bmatrix}\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n\mid\mathcal{U}_n)\\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n\mid\mathcal{U}_n)\\ \dots\dots\dots\dots\\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},m-1)}(\mathbf{t}_n\mid\mathcal{U}_n)\end{bmatrix},$$

and, consequently,

$$\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n,\alpha_{n+1})}(\mathbf{t}_{n+1}\mid\mathcal{U}_{n+1})=$$
$$=\sum_{\beta_n=0}^{m-1}a^{n+1}_{\alpha_{n+1}}(\beta_n)\mathbf{T}^{\mathbf{m^n}(\beta_n\oplus\alpha_n)}_{\mathbf{t}_n}\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},\beta_n)}(\mathbf{t}_n\mid\mathcal{U}_n).$$

Since $\mathbf{t}_{n+1}=(\mathbf{t}_n,t_{n+1})$, then believing $t_{n+1}=\alpha_n\underset{m}{\oplus}\beta_n$, we obtain:

$$\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n,\alpha_{n+1})}(\mathbf{t}_{n+1}\mid\mathcal{U}_{n+1})=\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n,\alpha_{n+1})}(\mathbf{t}_n,t_{n+1}\mid\mathcal{U}_{n+1})=$$
$$=\sum_{t_{n+1}=0}^{m-1}A^{n+1}_{\alpha_{n+1}}(\alpha_n\underset{m}{\oplus}t_{n+1})\mathbf{T}^{m^n t_{n+1}}_{\mathbf{t}_n}\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n\underset{m}{\oplus}t_{n+1})}(\mathbf{t}_n\mid\mathcal{U}_n)= \qquad (8)$$
$$=\sum_{t_{n+1}=0}^{m-1}A^{n+1}_{\alpha_{n+1}}(\alpha_n\underset{m}{\oplus}t_{n+1})\mathbf{T}^{m^n t_{n+1}}_{\mathbf{t}_n}\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n\underset{m}{\oplus}t_{n+1})}(\mathbf{t}_n+m^n t_{n+1}\mid\mathcal{U}_n).$$

So,

$$\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n,\alpha_{n+1})}(\mathbf{t}_n,t_{n+1}\mid\mathcal{U}_{n+1})=$$
$$=A^{n+1}_{\alpha_{n+1}}(\alpha_n\underset{m}{\oplus}t_{n+1})\cdot\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n\underset{m}{\oplus}t_{n+1})}(\mathbf{t}_n\mid\mathcal{U}_n). \qquad (9)$$

It is finally recurrent relation between $m$-complementary sequences of $\mathbf{G}^{[n+1]}_{m^{n+1}}\left[\mathcal{U}_{n+1}\right]$ and $\mathbf{G}^{[n]}_{m^n}\left[\mathcal{U}_n\right]$. From (9) we obtain expression for $\mathrm{com}^{[n+1]}_{\bar{\boldsymbol{\alpha}}_{n+1}}(\mathbf{t}_{n+1}\mid\mathcal{U}_{n+1})$:

$$\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n+1})}(\mathbf{t}_{n+1})=\prod_{s=1}^{n}A^{s+1}_{\alpha_{s+1}\underset{m}{\oplus}t_{s+2}}(\alpha_s\underset{m}{\oplus}t_{s+1}),\quad \alpha_0,t_{n+2}\equiv 0. \qquad (10)$$

In particular, for matrices in the form of the Fourier transform $\mathbf{U}^1_m=\mathbf{U}^2_m=\dots=\mathbf{U}^n_m=[\varepsilon^{\alpha t}_m]$ we have

$$\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n,\alpha_{n+1})}(\mathbf{t}_{n+1})=\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n,\alpha_{n+1})}(\mathbf{t}_n,t_{n+1})=$$
$$=\varepsilon_m^{\sum_{s=1}^{n}(\alpha_s\underset{m}{\oplus}t_{s+1})\left(\alpha_{s+1}\underset{m}{\oplus}t_{s+2}\right)}. \qquad (11)$$

Where $\alpha_0,t_{n+2}\equiv 0$. New sequences in (9) are orthogonal and $m$-complementary sequences.

## *Generalizations*

In this section, we introduce generalized $m$-complementary sequences. It is based on using new permutation matrices $\mathbf{P}^{\alpha_n}_m$ in (7). The mappings $g$: $\mathbf{X}\rightarrow\mathbf{X}$ of a set $\mathbf{X}$ into (or onto) itself are of particular importance. They form the following set $\mathbf{X}^{\mathbf{X}}:=\{g|g:\mathbf{X}\rightarrow\mathbf{X}\}$.

**Definition 2.** *One-to-one map from a set* $\mathbf{X}$ *to itself* $g$: $\mathbf{X}\rightarrow\mathbf{X}$, $x'=g(x)=g\circ x$ *is called a transformation of the set X.*

If $\mathbf{X}$ is finite and consists of $m$ elements (for example, $\mathbf{X}=\{0,1,2,\dots,m\}$) then a transformation of the set X is called a *permutation.* As is well known, the set of all permutations of $\mathbf{X}$ forms a group $S_m=Sum\{\mathbf{X}\}$ in which the product $\sigma\pi$ of a pair of permutations $\sigma,\pi$ is defined by $(\sigma\pi)\circ x:=\sigma\circ(\pi\circ x)$.

If $\mathbf{X}$ contains more than two elements, $S_m$ is not commutative. Any subgroup of $S_m$ is called a *permutation group* on $\mathbf{X}$, or a *group of permutations* of $\mathbf{X}$. We shall say that the permutations in $Sym$(X) *act* or *operate* on the elements of $\mathbf{X}$.

**Definition 3.** *A homomorphism of a group  on a set* $h$*:* $\mathbf{Gr}\rightarrow Sym\{\mathbf{X}\}$ *is called a permutation representation (or realization) of .*

The image $h(\mathbf{Gr})\subset Sym\{\mathbf{X}\}$ is a permutation group and the elements of  are represented as permutations of . A permutation representation is equivalent to an action of  on the set :  To specify an action, we need to define for element g$\in\mathbf{Gr}$ the corresponding permutation $h(g)$ of , that is, $h(g)\circ x$ for any $x\in\mathbf{X}$. We are going to write $h(g)\circ x$

in the short form $g \circ x$ and to call the group of transformations of . The pair $\langle \rangle$ is called a space with transformation group the elements $x \in \mathbf{X}$ are called points of the space .

**Definition 4.** *If is a permutation group of degree , then the permutation representation of is the linear permutation representation of :* $\mathbf{P}: \mathbf{Gr} \to \mathrm{GL}_m(\mathcal{A}lg)$ *which maps to the corresponding permutation matrix* $\mathbf{P}(g)$, *.*

That is, acts on by permuting the standard basis vectors $\{e_n\}_{n\in\mathbf{X}} \in \mathcal{A}lg^m$ such that

$$\mathbf{P}(g)e_n = e_{g\circ n} = e_{n'} \in \{e_n\}_{n\in\mathbf{X}},$$

where $\mathbf{P}(g)'s$ are the operators in $\mathcal{A}lg^m$ which define the above mentioned *linear* representation.

**Example 3**. Let

$$\mathbf{X} = [0,1,...,m-1],\ \mathbf{Gr} = \mathbf{Z}_m = \left\langle \{0,1,...,m-1\}, \underset{m}{\oplus} \right\rangle$$

be the cyclic group of order $m$. Then

$$\mathbf{P}(0) = \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix},\ \mathbf{P}(1) = \begin{bmatrix} & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ 1 & & & & \end{bmatrix},\ \mathbf{P}(2) = \begin{bmatrix} & & 1 & & \\ & & & 1 & \\ & & & & \ddots \\ 1 & & & & \\ & 1 & & & \end{bmatrix}, ..., \mathbf{P}(m-1) = \begin{bmatrix} & & & & 1 \\ 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 1 & \end{bmatrix}.$$

In particular, for $m=2$ and $m=3$we have

$$\mathbf{P}(0) = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix},\ \mathbf{P}(1) = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix};$$

$$\mathbf{P}(0) = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix},\ \mathbf{P}(1) = \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix},\ \mathbf{P}(2) = \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}.\ \square$$

In expression (7) was used *linear* permutation representation $\mathbf{P}(g)$ of only one group . However, we can use others finite groups of given order $m$. Let $\mathbf{Gr} = \mathbf{Gr}_m = \{g_\alpha\}_{\alpha=0}^{m-1}$ be a group of given order $m$ and $\{\mathbf{P}(g_\alpha)\}_{\alpha=0}^{m-1}$. Then

$$\mathbf{G}_{m^{n+1}}^{[n+1]}(\mathcal{U}_{n+1};\mathbf{Gr}_m) = \underset{\boldsymbol{\alpha}_n=0}{\overset{m^n-1}{\boxplus}} \begin{bmatrix} \mathrm{com}_{(\boldsymbol{\alpha}_n,0)}^{[n]}(\mathbf{t}_n \,|\, \mathcal{U}_{n+1};\mathbf{Gr}_m) \\ \mathrm{com}_{(\boldsymbol{\alpha}_n,1)}^{[n]}(\mathbf{t}_n \,|\, \mathcal{U}_{n+1};\mathbf{Gr}_m) \\ \cdots\cdots\cdots\cdots \\ \mathrm{com}_{(\boldsymbol{\alpha}_n,m-1)}^{[n]}(\mathbf{t}_n \,|\, \mathcal{U}_{n+1};\mathbf{Gr}_m) \end{bmatrix} =$$

$$= \underset{\boldsymbol{\alpha}_n=0}{\overset{m^n-1}{\boxplus}} \mathbf{U}_m^{n+1} \cdot \left( \mathbf{P}_m(g_{\alpha_n}) \cdot \begin{bmatrix} \mathbf{I}_{\mathbf{t}_n} & & & \\ & \mathbf{T}_{\mathbf{t}_n}^{1\cdot m^n} & & \\ & & \ddots & \\ & & & \mathbf{T}_{\mathbf{t}_n}^{(m-1)\cdot m^n} \end{bmatrix} \cdot \tilde{\mathbf{P}}_m(g_{\alpha_n}) \right) \cdot \begin{bmatrix} \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n \,|\, \mathcal{U}_n;\mathbf{Gr}_m) \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n \,|\, \mathcal{U}_n;\mathbf{Gr}_m) \\ \cdots\cdots\cdots\cdots \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},m-1)}^{[n]}(\mathbf{t}_n \,|\, \mathcal{U}_n;\mathbf{Gr}_m) \end{bmatrix} \quad (12)$$

is the Golay matrix associated with triple $\left(\mathbf{Gr}_m, \{\mathbf{U}_m^1, \mathbf{U}_m^2, ..., \mathbf{U}_m^{n+1}\}, \mathcal{A}lg\right)$.

**Example 4.** For $m=4$ we have two groups: $\mathbf{Z}_4 = \{0,1,2,3\}$ and $\mathbf{Z}_2 \times \mathbf{Z}_2 = \{(0,0),(0,1),(1,0),(1,1)\}$. For both groups we have the following permutation representations:

$$\mathbf{P}(0) = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix},\ \mathbf{P}(1) = \begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ 1 & & & \end{bmatrix},\ \mathbf{P}(2) = \begin{bmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{bmatrix},\ \mathbf{P}(3) = \begin{bmatrix} & & & 1 \\ 1 & & & \\ & 1 & & \\ & & 1 & \end{bmatrix},$$

$$\mathbf{P}(0,0) = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix},\ \mathbf{P}(0,1) = \begin{bmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{bmatrix},\ \mathbf{P}(1,0) = \begin{bmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{bmatrix},\ \mathbf{P}(1,1) = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{bmatrix}.$$

Hence, we can construct two different set of Golay matrices associated with two triples

1) $\left(\mathbf{Z}_4, \{\mathbf{U}_m^1, \mathbf{U}_m^2, ..., \mathbf{U}_m^{n+1}\}, \mathcal{A}lg\right)$,

2) $\left(\mathbf{Z}_2 \times \mathbf{Z}_2, \{\mathbf{U}_m^1, \mathbf{U}_m^2, ..., \mathbf{U}_m^{n+1}\}, \mathcal{A}lg\right)$,

respectively. □

Let $\mathcal{G}_{n+1} := \left\{ \mathbf{Gr}_m^1, \mathbf{Gr}_m^2, ..., \mathbf{Gr}_m^n, \mathbf{Gr}_m^{n+1} \right\} = \left\{ \mathcal{G}_m^n, \mathbf{Gr}_m^{n+1} \right\}$ be a set of arbitrary groups of given order $m$: $\mathbf{Gr}_m^1 = \left\{ g_{\alpha_1}^1 \right\}_{\alpha_1=0}^{m-1}, ..., \mathbf{Gr}_m^{n+1} = \left\{ g_{\alpha_{n+1}}^1 \right\}_{\alpha_{n+1}=0}^{m-1}$. Then we can use on each $k^{th}$iteration permutation representations $\left\{ \mathbf{P}_m^k(g_{\alpha_k}) \right\}_{\alpha_{k=0}}^{m-1}$ for $\mathbf{Gr}_m^k$. In this case, we obtain the following Golay transform

$$\mathbf{G}_{m^{n+1}}^{[n+1]}(\mathcal{U}_{n+1};\mathcal{G}_{n+1}) = \boxplus_{\boldsymbol{\alpha}_n=0}^{m^n-1} \begin{bmatrix} \mathrm{com}_{(\boldsymbol{\alpha}_n,0)}^{[n]}(\mathbf{t}_n \mid \mathcal{U}_{n+1};\mathcal{G}_{n+1}) \\ \mathrm{com}_{(\boldsymbol{\alpha}_n,1)}^{[n]}(\mathbf{t}_n \mid \mathcal{U}_{n+1};\mathcal{G}_{n+1}) \\ \dots\dots\dots\dots \\ \mathrm{com}_{(\boldsymbol{\alpha}_n,m-1)}^{[n]}(\mathbf{t}_n \mid \mathcal{U}_{n+1};\mathcal{G}_{n+1}) \end{bmatrix} =$$

$$= \boxplus_{\boldsymbol{\alpha}_n=0}^{m^n-1} \mathbf{U}_m^{n+1} \cdot \left( \mathbf{P}_m^{n+1}(g_{\alpha_n}) \cdot \begin{bmatrix} \mathbf{I}_{\mathbf{t}_n} & & & \\ & \mathbf{T}_{\mathbf{t}_n}^{1\cdot m^n} & & \\ & & \ddots & \\ & & & \mathbf{T}_{\mathbf{t}_n}^{(m-1)\cdot m^n} \end{bmatrix} \cdot \tilde{\mathbf{P}}_m^{n+1}(g_{\alpha_n}) \right) \cdot \begin{bmatrix} \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n \mid \mathcal{U}_n;\mathcal{G}_n) \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n \mid \mathcal{U}_n;\mathcal{G}_n) \\ \dots\dots\dots\dots \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},m-1)}^{[n]}(\mathbf{t}_n \mid \mathcal{U}_n;\mathcal{G}_n) \end{bmatrix}. \quad (13)$$

It is associated with triple

$$\left( \left\{ \mathbf{Gr}_m^1, \mathbf{Gr}_m^2, ..., \mathbf{Gr}_m^{n+1} \right\}, \left\{ \mathbf{U}_m^1, \mathbf{U}_m^2, ..., \mathbf{U}_m^{n+1} \right\}, \mathcal{A}lg \right).$$

### *Fast Golay transforms*

Let us consider expressions (8) and (9) for $m=2$ (*i.e.,* expressions (6) and (7) from our work [7]):

$$\mathrm{com}_{(\boldsymbol{\alpha}_{n+1})}^{[n+1]}(\mathbf{t}_{n+1}) = \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},\alpha_n,\alpha_{n+1})}^{[n+1]}(\mathbf{t}_n, t_{n+1}) = \\ = \sum_{t_{n+1}=0}^{1} (-1)^{\left(\alpha_n \oplus_2 t_{n+1}\right)\alpha_{n+1}} \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},\alpha_n \oplus_2 t_{n+1})}^{[n]} \left( \mathbf{t}_n + 2^n \cdot t_{n+1} \right), \quad (14)$$

$$\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},\alpha_n,\alpha_{n+1})}^{[n+1]}(\mathbf{t}_n, t_{n+1}) = (-1)^{\left(\alpha_n \oplus_2 t_{n+1}\right)\alpha_{n+1}} \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},\alpha_n \oplus_2 t_{n+1})}^{[n]} \times \\ \times \left( \mathbf{t}_n \right) = (-1)^{\alpha_n\alpha_{n+1}} (-1)^{\alpha_{n+1}t_{n+1}} \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},\alpha_n \oplus_2 t_{n+1})}^{[n]} \left( \mathbf{t}_n \right) \quad (15)$$

and find matrix representations of these expressions. We introduce the following σ-parametrized $(2^n \times 2^n)$-matrix:

$${}^{\sigma}\mathbf{G}_{2^n}^{[n]} := \boxplus_{\boldsymbol{\alpha}_n=0}^{2^n-1} P_2^{\sigma} \begin{bmatrix} \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n) \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n) \end{bmatrix} = \begin{cases} {}^{0}\mathbf{G}_{2^n}^{[n]} = \boxplus_{\boldsymbol{\alpha}_n=0}^{2^n-1} \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} \begin{bmatrix} \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n) \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n) \end{bmatrix}, & \sigma = 0, \\ {}^{1}\mathbf{G}_{2^n}^{[n]} = \boxplus_{\boldsymbol{\alpha}_n=0}^{2^n-1} \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \begin{bmatrix} \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n) \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n) \end{bmatrix}, & \sigma = 1; \end{cases}$$

$$= \begin{cases} {}^{0}\mathbf{G}_{2^n}^{[n]} = \boxplus_{\boldsymbol{\alpha}_n=0}^{2^n-1} P_2^{0} \begin{bmatrix} \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n) \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n) \end{bmatrix}, & \sigma = 0 \\ {}^{1}\mathbf{G}_{2^n}^{[n]} = \boxplus_{\boldsymbol{\alpha}_n=0}^{2^n-1} P_2^{1} \begin{bmatrix} \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n) \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n) \end{bmatrix}, & \sigma = 1 \end{cases} = \begin{cases} {}^{0}\mathbf{G}_{2^n}^{[n]} = \boxplus_{\boldsymbol{\alpha}_n=0}^{2^n-1} \begin{bmatrix} \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n) \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n) \end{bmatrix}, & \sigma = 0, \\ {}^{1}\mathbf{G}_{2^n}^{[n]} = \boxplus_{\boldsymbol{\alpha}_n=0}^{2^n-1} \begin{bmatrix} \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n) \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n) \end{bmatrix}, & \sigma = 1, \end{cases}$$

and construct the direct sum of introduced matrices

$$\tilde{\mathbf{G}}_{2^{n+1}}^{[n+1]} = \bigoplus_{\sigma=0}^{1} {}^{(\sigma)}\mathbf{G}_{2^n}^{[n]} = \left[ \begin{array}{c:c} {}^{(0)}\mathbf{G}_{2^n}^{[n]} & \\ \hdashline & {}^{(1)}\mathbf{G}_{2^n}^{[n]} \end{array} \right] = \left[ \begin{array}{c:c} \left( \mathbf{I}_{2^{n-1}} \otimes \mathbf{P}_2^0 \right) \mathbf{G}_{2^n}^{[n]} & \\ \hdashline & \left( \mathbf{I}_{2^{n-1}} \otimes \mathbf{P}_2^1 \right) \mathbf{G}_{2^n}^{[n]} \end{array} \right] =$$

$$= \left[ \begin{array}{c:c} \boxplus_{\boldsymbol{\alpha}_{n-1}=0}^{2^{n-1}-1} \begin{bmatrix} \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n) \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n) \end{bmatrix} & \\ \hdashline & \boxplus_{\boldsymbol{\alpha}_{n-1}=0}^{2^{n-1}-1} \begin{bmatrix} \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n + 2^n) \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n + 2^n) \end{bmatrix} \end{array} \right]. \quad (16)$$

From (16) we see that $\tilde{\mathbf{G}}_{2^{n+1}}^{[n+1]}$ represents $\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},\alpha_n \oplus_2 t_{n+1})}^{[n]} \left( \mathbf{t}_n + 2^n \cdot t_{n+1} \right)$ in (14). It is easy to see, that

$$\tilde{\mathbf{G}}_{2^{n+1}}^{[n+1]} = \left[ \begin{array}{c:c} \left[ \mathbf{I}_{2^{n-1}} \otimes \mathbf{P}_2^0 \right] & \\ \hdashline & \left[ \mathbf{I}_{2^{n-1}} \otimes \mathbf{P}_2^1 \right] \end{array} \right] \times \left[ \begin{array}{c:c} \mathbf{G}_{2^n}^{[n]} & \\ \hdashline & \mathbf{G}_{2^n}^{[n]} \end{array} \right] == \left[ \delta_{\alpha_{n+1}}^{(2)}(t_{n+1}) \left[ \mathbf{I}_{2^{n-1}} \otimes \mathbf{P}_2^{t_{n+1}} \right] \right] \times \left[ \mathbf{I}_2 \otimes \mathbf{G}_{2^n}^{[n]} \right] = \mathbf{P}_2^{\{t_{n+1}\}} \cdot \left[ \mathbf{I}_2 \otimes \mathbf{G}_{2^n}^{[n]} \right],$$

where

$$
\mathbf{P}_{2^{n+1}}^{\{t_{n+1}\}} := \left[\delta_{\alpha_{n+1}}^{(2)}(t_{n+1})\left[\mathbf{I}_{2^{n-1}} \otimes \mathbf{P}_2^{t_{n+1}}\right]\right] = \left[\begin{array}{c:c} \left[\mathbf{I}_{2^{n-1}} \otimes \mathbf{P}_2^0\right] & \\ \hdashline & \left[\mathbf{I}_{2^{n-1}} \otimes \mathbf{P}_2^1\right] \end{array}\right]
$$

is the permutation matrix with controlling digit $\{t_{n+1}\}$. According to (15) the Golay matrix $\mathbf{G}_{2^{n+1}}^{[n+1]}$ is the product of three matrices

$$
\begin{gathered}
\mathbf{G}_{2^{n+1}}^{[n+1]} = \Delta\left\{(-1)^{\alpha_n \alpha_{n+1}}\right\}\left[\delta_{\boldsymbol{\alpha}_n, \mathbf{t}_n}^{(2^n)}(-1)^{\alpha_{n+1} t_{n+1}}\right]\tilde{\mathbf{G}}_{2^{n+1}}^{[n+1]} = \Delta\left\{(-1)^{\alpha_n \alpha_{n+1}}\right\}\left[\delta_{\boldsymbol{\alpha}_n, \mathbf{t}_n}^{(2^n)}(-1)^{\alpha_{n+1} t_{n+1}}\right], \\
\left[\delta_{\alpha_{n+1}}^{(2)}(t_{n+1})\left[\mathbf{I}_{2^{n-1}} \otimes \mathbf{P}_2^{t_{n+1}}\right]\right]\left[\mathbf{I}_2 \otimes \mathbf{G}_{2^n}^{[n]}\right] = \Delta\left\{(-1)^{\alpha_n \alpha_{n+1}}\right\}\left[\delta_{\boldsymbol{\alpha}_n, \mathbf{t}_n}^{(2^n)}(-1)^{\alpha_{n+1} t_{n+1}}\right]\mathbf{P}_{2^{n+1}}^{\{t_{n+1}\}}\left[\mathbf{I}_2 \otimes \mathbf{G}_{2^n}^{[n]}\right].
\end{gathered} \tag{17}
$$

Where $\Delta\{(-1)^{\alpha_n \alpha_{n+1}}\} = \mathbf{diag}\{(-1)^{\alpha_n \alpha_{n+1}}\}$ is diagonal matrix, and $\left[\delta_{\boldsymbol{\alpha}_n, \mathbf{t}_n}^{(2^n)}(-1)^{\alpha_{n+1} t_{n+1}}\right]$ has the following structure

$$
\left[\delta_{\boldsymbol{\alpha}_n, \mathbf{t}_n}^{(2^n)}(-1)^{\alpha_{n+1} t_{n+1}}\right] = \left[\mathbf{I}_{2^n} \otimes \begin{bmatrix} 1 \\ 1 \end{bmatrix} \middle| \mathbf{I}_{2^n} \otimes \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right] = \left[\mathbf{I}_{2^n} \middle| \mathbf{I}_{2^n}\right] \hat{\otimes} \left[\begin{array}{c|c} 1 & 1 \\ 1 & -1 \end{array}\right] =
$$

$$
= \left[\left[\delta_{\boldsymbol{\alpha}_n, \mathbf{t}_n}^{(2^n)}\right] \otimes \begin{bmatrix} 1 \\ 1 \end{bmatrix} \middle| \left[\delta_{\boldsymbol{\alpha}_n, \mathbf{t}_n}^{(2^n)}\right] \otimes \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right] = \begin{array}{c} \\ \left\{\begin{array}{c} \alpha_{n+1} = 0 \\ \alpha_{n+1} = 1 \\ \hline \alpha_{n+1} = 0 \\ \alpha_{n+1} = 1 \\ \hline \alpha_{n+1} = 0 \\ \alpha_{n+1} = 1 \\ \hline \alpha_{n+1} = 0 \\ \alpha_{n+1} = 1 \end{array}\right. \end{array} \begin{array}{c} \begin{array}{ccc:ccc} & t_{n+1} = 0 & & & t_{n+1} = 1 & \end{array} \\ \left[\begin{array}{ccc|ccc} 1 & & & 1 & & \\ 1 & & & -1 & & \\ \hdashline & 1 & & & 1 & \\ & 1 & & & -1 & \\ \hdashline & \vdots & & & \vdots & \\ & \vdots & & & \vdots & \\ \hdashline & & 1 & & & 1 \\ & & 1 & & & -1 \end{array}\right] \end{array} := \mathbb{N}_{2^{n+1}}. \tag{18}
$$

Here $\hat{\otimes}$ is new tensor product:

$$
\left[\mathbf{I}_{2^n} \middle| \mathbf{I}_{2^n}\right] \hat{\otimes} \left[\begin{array}{c|c} 1 & 1 \\ 1 & -1 \end{array}\right] := \left[\mathbf{I}_{2^n} \otimes \begin{bmatrix} 1 \\ 1 \end{bmatrix} \middle| \mathbf{I}_{2^n} \otimes \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right].
$$

From recurrent relation (17) we obtain

$$
\begin{gathered}
\mathbf{G}_{2^n}^{[n]} = \left(\prod_{k=2}^{n}\left[\mathbf{I}_{2^{n-k}} \otimes \Delta_{2^k} \cdot \mathbb{N}_{2^k} \cdot \mathbf{P}_{2^k}^{\{t_k\}}\right]\right) \cdot \left[\mathbf{I}_{2^{n-k+1}} \otimes \mathbf{G}_{2^1}^{[1]}\right] = \prod_{k=2}^{n}\left\{\mathbf{I}_{2^{n-k}} \otimes \left[\Delta\left\{(-1)^{\alpha_{k-1}\alpha_k}\right\}\right] \cdot \left[\delta_{\boldsymbol{\alpha}_{k-1}, \mathbf{t}_{k-1}}^{(2^{k-1})}(-1)^{\alpha_k t_k}\right] \cdot\right. \\
\left. \cdot \left[\delta_{\alpha_k}^{(2)}(t_k)\left[\mathbf{I}_{2^{k-2}} \otimes \mathbf{P}_2^{t_k}\right]\right]\right\} \cdot \left[\mathbf{I}_{2^{n-k+1}} \otimes \mathbf{G}_{2^1}^{[1]}\right].
\end{gathered} \tag{19}
$$

This expression represents the fast algorithm for the Golay transform.

**Example 5.**

$$
\mathbf{G}_{2^2}^{[2]} = \begin{bmatrix} \mathrm{com}_{(0,0)}^{[2]}(\mathbf{t}_2) \\ \mathrm{com}_{(0,1)}^{[2]}(\mathbf{t}_2) \\ \mathrm{com}_{(1,0)}^{[2]}(\mathbf{t}_2) \\ \mathrm{com}_{(1,1)}^{[2]}(\mathbf{t}_2) \end{bmatrix} = \left[\begin{array}{cccc} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ \hdashline 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{array}\right] = \left[\begin{array}{cc:cc} 1 & & & \\ & 1 & & \\ \hdashline & & 1 & \\ & & & -1 \end{array}\right] \left[\begin{array}{cc:cc} 1 & & 1 & \\ 1 & & -1 & \\ \hdashline & 1 & & 1 \\ & 1 & & -1 \end{array}\right] \left[\begin{array}{cc:cc} 1 & & & \\ & 1 & & \\ \hdashline & & & 1 \\ & & 1 & \end{array}\right] \left[\begin{array}{cc:cc} 1 & 1 & & \\ 1 & -1 & & \\ \hdashline & & 1 & 1 \\ & & 1 & -1 \end{array}\right] =
$$

$$
= \left[\mathbf{I}_{2^0} \otimes \Delta_{2^2} \cdot \mathbb{N}_{2^2} \cdot \mathbf{P}_{2^2}^{\{t_2\}}\right] \cdot \left[\mathbf{I}_{2^1} \otimes \mathbf{G}_{2^1}^{[1]}\right].
$$

$$
\mathbf{G}_{2^3}^{[3]} = \begin{bmatrix} \mathrm{com}_{(0,0,0)}^{[3]}(\mathbf{t}_3) \\ \mathrm{com}_{(0,0,1)}^{[3]}(\mathbf{t}_3) \\ \mathrm{com}_{(0,1,0)}^{[3]}(\mathbf{t}_3) \\ \mathrm{com}_{(0,1,1)}^{[3]}(\mathbf{t}_3) \\ \mathrm{com}_{(1,0,0)}^{[3]}(\mathbf{t}_3) \\ \mathrm{com}_{(1,0,1)}^{[3]}(\mathbf{t}_3) \\ \mathrm{com}_{(1,1,0)}^{[3]}(\mathbf{t}_3) \\ \mathrm{com}_{(1,1,1)}^{[3]}(\mathbf{t}_3) \end{bmatrix} = \left[\begin{array}{cccc:cccc} 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 \\ \hdashline 1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 \\ -1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 \\ \hdashline 1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 \\ \hdashline -1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 \end{array}\right] = \left[\begin{array}{cccc:cccc} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ & & & -1 & & & & \\ \hdashline & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & -1 & \\ & & & & & & & 1 \end{array}\right] \cdot \left[\begin{array}{cc:cc:cc:cc} 1 & & & & 1 & & & \\ 1 & & & & -1 & & & \\ \hdashline & 1 & & & & 1 & & \\ & 1 & & & & -1 & & \\ \hdashline & & 1 & & & & 1 & \\ & & 1 & & & & -1 & \\ \hdashline & & & 1 & & & & 1 \\ & & & 1 & & & & -1 \end{array}\right] \times
$$

$$
\times\begin{bmatrix} 1&&&&&&& \\ &1&&&&&& \\ &&1&&&&& \\ &&&1&&&& \\ &&&&&1&& \\ &&&&1&&& \\ &&&&&&&1 \\ &&&&&&1& \end{bmatrix}\cdot\begin{bmatrix} 1&1&1&-1&&&& \\ 1&1&-1&1&&&& \\ 1&-1&1&1&&&& \\ -1&1&-1&1&&&& \\ &&&&1&1&1&-1 \\ &&&&1&1&-1&1 \\ &&&&1&-1&1&1 \\ &&&&-1&1&1&1 \end{bmatrix}=\begin{bmatrix} 1&&&&&&& \\ &1&&&&&& \\ &&1&&&&& \\ &&&-1&&&& \\ &&&&1&&& \\ &&&&&1&& \\ &&&&&&-1& \\ &&&&&&&1 \end{bmatrix}\times
$$

$$
\times\begin{bmatrix} 1&&&&1&&& \\ 1&&&&-1&&& \\ &1&&&&1&& \\ &1&&&&-1&& \\ &&1&&&&1& \\ &&1&&&&-1& \\ &&&1&&&&1 \\ &&&1&&&&-1 \end{bmatrix}\cdot\begin{bmatrix} 1&&&&&&& \\ &1&&&&&& \\ &&1&&&&& \\ &&&1&&&& \\ &&&&&1&& \\ &&&&1&&& \\ &&&&&&&1 \\ &&&&&&1& \end{bmatrix}\cdot\begin{bmatrix} 1&&&&&&& \\ &1&&&&&& \\ &&1&&&&& \\ &&&-1&&&& \\ &&&&1&&& \\ &&&&&1&& \\ &&&&&&1& \\ &&&&&&&-1 \end{bmatrix}\times
$$

$$
\times\begin{bmatrix} 1&&1&&&&& \\ 1&&-1&&&&& \\ &1&&1&&&& \\ &1&&-1&&&& \\ &&&&1&&1& \\ &&&&1&&-1& \\ &&&&&1&&1 \\ &&&&&1&&-1 \end{bmatrix}\cdot\begin{bmatrix} 1&&&&&&& \\ &1&&&&&& \\ &&&1&&&& \\ &&1&&&&& \\ &&&&1&&& \\ &&&&&1&& \\ &&&&&&&1 \\ &&&&&&1& \end{bmatrix}\cdot\begin{bmatrix} 1&1&&&&&& \\ 1&-1&&&&&& \\ &&1&1&&&& \\ &&1&-1&&&& \\ &&&&1&1&& \\ &&&&1&-1&& \\ &&&&&&1&1 \\ &&&&&&1&-1 \end{bmatrix}=
$$

$$
=\left[\mathbf{I}_{2^0}\otimes\Delta_{2^3}\cdot\mathbb{N}_{2^3}\cdot\mathbf{P}_{2^3}^{\{t_3\}}\right]\cdot\left[\mathbf{I}_{2^1}\otimes\Delta_{2^2}\cdot\mathbb{N}_{2^2}\cdot\mathbf{P}_{2^2}^{\{t_2\}}\right]\cdot\left[\mathbf{I}_{2^2}\otimes\mathbf{G}_{2^1}^{[1]}\right]. \qquad \square
$$

### *Conclusion and future researches*

In this paper, we have shown a new unified approach to the so-called generalized multi-parameter $m$–complementary sequences. The approach is based on a new iteration generating construction. This construction has a rich algebraic structure. It is associated not with the triple $(\mathbf{Z}_2, \mathcal{F}_2, \mathbf{C})$, but with

1) $(\mathbf{Z}_m, \mathbf{U}_m, \mathcal{A}lg)$,

2) $\left(\mathbf{Z}_m, \left\{\mathbf{U}_m^1, \mathbf{U}_m^2, ..., \mathbf{U}_m^n\right\}, \mathcal{A}lg\right)$,

3) $\left(\mathbf{Gr}_m, \left\{\mathbf{U}_m^1, \mathbf{U}_m^2, ..., \mathbf{U}_m^n\right\}, \mathcal{A}lg\right)$ or with

4) $\left(\left\{\mathbf{Gr}_m^1, \mathbf{Gr}_m^2, ..., \mathbf{Gr}_m^n\right\}, \left\{\mathbf{U}_m^1, \mathbf{U}_m^2, ..., \mathbf{U}_m^n\right\}, \mathcal{A}lg\right)$,

where $\left\{\mathbf{U}_m^1, \mathbf{U}_m^2, ..., \mathbf{U}_m^n\right\}$ is a set of arbitrary unitary $(m\times m)$ -transforms and $\left\{\mathbf{Gr}_m^1, \mathbf{Gr}_m^2, ..., \mathbf{Gr}_m^n\right\}$ is a set of arbitrary groups of given order $m$. Furthermore, we have derived demonstrated fast algorithms for Golay transforms.

We are going to use generalized multi-parameter $m$-complementary sequences as subcarriers of Intelligent OFDM telecommunication system. Most of the data transmission systems nowadays use orthogonal frequency division multiplexing telecommunication system (OFDM-TCS) based on the discrete Fourier transform (DFT) $\mathcal{F}_N$. The conventional OFDM will be denoted by the symbol $\mathcal{F}_N$-**OFDM**. Conventional OFDM-TCS makes use of signal orthogonality of the multiple sub-carriers $e^{j2\pi kn/N}$ (complex exponential harmonics). Sub-carriers $\left\{\mathbf{subc}_k(n)\right\}_{k=0}^{N-1}=\left\{e^{j2\pi kn/N}\right\}_{k=0}^{N-1}$ form matrix of DFT $\mathcal{F}_N=\left[\mathbf{subc}_k(n)\right]_{k,n=0}^{N-1}\equiv\left[e^{j2\pi kn/N}\right]_{k,n=0}^{N-1}$.

At the time, the idea of using the fast algorithm of different orthogonal transforms $\mathbf{U}_N=\left[\mathbf{subc}_k(n)\right]_{k,n=0}^{N-1}$ for a software-based implementation of the OFDM's modulator and demodulator, transformed this technique from an attractive, but difficult to implement idea, into an incredibly successful story of the data transmission. OFDM-TCS, based on arbitrary orthogonal (unitary) transform $\mathbf{U}_N$ will be denoted as $\mathbf{U}_N$-**OFDM**. The idea which links $\mathcal{F}_N$-**OFDM** and $\mathbf{U}_N$-**OFDM** is that, in the same manner that the complex exponentials $\left\{e^{j2\pi kn/N}\right\}_{k=0}^{N-1}$ are orthogonal to each-other, the members of a family of $\mathbf{U}_N$-sub-carriers $\left\{\mathbf{subc}_k(n)\right\}_{k=0}^{N-1}$ (rows of the matrix $\mathcal{U}_N$) will satisfy the same property. The $\mathbf{U}_N$-**OFDM** reshapes the multi-carrier transmission concept, by using carriers $\left\{\mathbf{subc}_k(n)\right\}_{k=0}^{N-1}$ instead of OFDM's complex exponentials $\left\{e^{j2\pi kn/N}\right\}_{k=0}^{N-1}$. In this paper, we propose a simple and effective anti-eavesdropping and anti-jamming Intelligent OFDM system, based on MPTs. In our Intelligent-OFDM-TCS we are going to use multi-parameter Golay transform $\mathbf{G}_{2^n}(\varphi_1, \varphi_2,\ldots, \varphi_q)$ at the place of DFT $\mathcal{F}_N$. We are going to study of Intell- $\mathbf{G}_{2^n}(\varphi_1, \varphi_2,\ldots, \varphi_q)$-OFDM-TCS to find out optimal values of parameters optimized PARP, BER, SER, anti-eavesdropping and anti-jamming effects.

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